

Indecomposable branched coverings over the projective plane by surfaces M with $\chi(M) \leq 0$

Natalia A. Viana Bedoya, Daciberg Lima Gonçalves and
Elena A. Kudryavtseva

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Abstract

In this work we study the decomposability property of branched coverings of degree d odd, over the projective plane, where the covering surface has Euler characteristic ≤ 0 . The latter condition is equivalent to say that the defect of the covering is greater than d . We show that, given a datum $\mathcal{D} = \{D_1, \dots, D_s\}$ with an even defect greater than d , it is realizable by an indecomposable branched covering over the projective plane. The case when d is even is known.

Key words: branched coverings, primitive groups, Hurwitz problem, permutation groups, projective plane.

Introduction

A branched covering $\phi : M \longrightarrow N$ of degree d between closed surfaces determines a finite collection \mathcal{D} of partitions of d , the branch datum of degree d , in 1 – 1 correspondence with the branch point set $B_\phi \subset N$. The *total defect* of \mathcal{D} is defined by $\nu(\mathcal{D}) = \sum_{x \in B_\phi} (d - \#\phi^{-1}(x))$. Given a collection of partitions \mathcal{D} of d and $N \neq S^2, \mathbb{R}P^2$, there are necessary and sufficient conditions on \mathcal{D} to realize it as branch datum of a branched covering of degree d over N with a connected covering surface, for more details see [8], [9], [11]. Such collections are called either *admissible data* if $\chi(N) \leq 0$, or *nonorientable-admissible* if $N = \mathbb{R}P^2$ and the covering surface is nonorientable. Due to [8], these conditions are either $\nu(\mathcal{D}) \equiv 0 \pmod{2}$ if $\chi(N) \leq 0$, or $d - 1 \leq \nu(\mathcal{D}) \equiv 0 \pmod{2}$ if $N = \mathbb{R}P^2$ and the covering surface is nonorientable, see (4) for $N = \mathbb{R}P^2$.

Decomposability properties of branched coverings between surfaces provide three classes of admissible data: those data that are realizable only by decomposable branched coverings, those that are realizable only by indecomposable branched

coverings and those that are realizable by both, decomposable and indecomposable branched coverings. A datum realizable by a decomposable primitive branched covering over N with a connected covering surface is called *decomposable primitive datum on N* . A characterization of the admissible data which are realizable by decomposable primitive branched coverings over $N \neq S^2$ is known. This characterization follows from [2, Proposition 2.6] for $\chi(N) \leq 0$, and [1] for $N = \mathbb{R}P^2$. The proof for $\chi(N) = 1$, which is in [1], is similar to the proof of Proposition 2.6 in [2].

Proposition 0.1 (Proposition 2.6 [2]). *Admissible data \mathcal{D} are decomposable primitive on N , with $\chi(N) \leq 0$, if and only if there exists a factorization of \mathcal{D} such that its left factor is a non-trivial admissible datum.* \square

The question of realization by indecomposable branched coverings of an admissible datum is interesting for decomposable data, otherwise clearly the problem has a positive solution. This question has been completely solved in [2] for the case where N is a closed surface with $\chi(N) \leq 0$. Namely:

Theorem (Theorem 3.3 [2]). *Every non-trivial admissible data are realized on any N , with $\chi(N) \leq 0$, by an indecomposable (and hence primitive) branched covering.* \square

In this case, there does not exist an admissible datum realizable only by decomposable branched coverings, therefore decomposable and indecomposable realizations coexist (whenever a decomposable realization exists) for the same data.

It remains to study the “indecomposability” of branched coverings between surfaces where $\chi(N) = 1$, i.e. $N = \mathbb{R}P^2$, since we are not considering the case $N = S^2$. So let $M \rightarrow \mathbb{R}P^2$ be a branched covering of degree d . The even degree case, i.e. when d is even, has been solved in [3]. The result is:

Theorem (Theorem 3.7 [3]). *Let d be even and $\mathcal{D} = \{D_1, \dots, D_s\}$ a nonorientable-admissible datum (see (4) or theorem 1.6) such that $s > 0$ and $D_i \neq [1, \dots, 1]$ for any $i \in \{1, \dots, s\}$. Then \mathcal{D} is realizable by an indecomposable branched covering over $\mathbb{R}P^2$ if and only if at least one of the following conditions holds:*

- (1) $d = 2$, or
- (2) there is $i \in \{1, \dots, s\}$ such that $D_i \neq [2, \dots, 2]$, or
- (3) $d > 4$ and $s > 2$.

\square

Except for the case $d = 2$, where the branched covering clearly is never decomposable, the theorem above together with the analogue for $N = \mathbb{R}P^2$ of Proposition 2.6 [2] characterize nonorientable-admissible data realizable by both, decomposable primitive and indecomposable branched coverings of even degree over $\mathbb{R}P^2$. Moreover nonorientable-admissible data like either $\{[2, \dots, 2], [2, \dots, 2]\}$ or

$\{[2, 2], [2, 2], \dots, [2, 2]\}$ with $d > 2$ are realizable only by decomposable branched coverings.

In this work we study the case of odd degree with $N = \mathbb{R}P^2$ and $\nu(\mathcal{D}) > d - 1$ (i.e. $\chi(M) \leq 0$, compare (4)). Our main result is:

Theorem 3.1. *Let \mathcal{D} be a collection of partitions of an odd integer d such that $d - 1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$ (compare (4)). Then it can be realized as the branch datum of an indecomposable (and hence primitive) branched covering of degree d over the projective plane with a connected covering surface.*

Our technique of proving this theorem will allow us to show that certain collections of partitions of an odd integer d are realizable as branch data of branched coverings over the 2-sphere with a connected covering surface (see Theorem 4.7).

1 Preliminaries, terminology and notation

1.1 Permutation groups

We denote by $\Sigma_d = \text{Sym}_\Omega$ the symmetric group on a set Ω with d elements and by 1_d its identity element. If $\alpha \in \Sigma_d$ and $x \in \Omega$, x^α is the image of x by α . An explicit permutation α will be written sometimes as a product of disjoint cycles, i.e. its *cyclic decomposition*. The set of lengths of the cycles in the cyclic decomposition of α , including the trivial ones, defines a partition of d , say $D_\alpha = [d_{\alpha_1}, \dots, d_{\alpha_t}]$, called *the cyclic structure of α* . Define $\nu(\alpha) := \sum_{i=1}^t (d_{\alpha_i} - 1)$, then α will be an *even permutation* if $\nu(\alpha) \equiv 0 \pmod{2}$. Given a partition D of d , we say $\alpha \in D$ if the cyclic structure of α is D and we put $\nu(D) := \nu(\alpha)$.

For $1 < r \leq d$, a permutation $\alpha \in \Sigma_d$ is called a *r-cycle* if, in its cyclic decomposition, its unique non-trivial cycle has length r . Permutations $\alpha, \beta \in \Sigma_d$ are *conjugate* if there is $\lambda \in \Sigma_d$ such that $\alpha^\lambda := \lambda\alpha\lambda^{-1} = \beta$. It is a known fact that conjugate permutations have the same cyclic structure.

Given a permutation group G on Ω and $x \in \Omega$, one defines the *isotropy subgroup of x* , $G_x := \{g \in G : x^g = x\}$, and the *orbit of x by G* , $x^G := \{x^g : g \in G\}$. For $H \subset G$, the subsets $\text{Supp}(H) := \{x \in \Omega : x^h \neq x \text{ for some } h \in H\}$ and $\text{Fix}(H) := \{x \in \Omega : x^h = x \text{ for all } h \in H\}$ are defined. For $\Lambda \subset \Omega$ and $g \in G$, $\Lambda^g := \{y^g : y \in \Lambda\}$.

G is said *transitive* if for all $x, y \in \Omega$ there is $g \in G$ such that $x^g = y$. A non-empty subset $\Lambda \subset \Omega$ is a *block* of a transitive G if for each $g \in G$ either $\Lambda^g = \Lambda$ or $\Lambda^g \cap \Lambda = \emptyset$. A block Λ is *trivial* if either $\Lambda = \Omega$ or $\Lambda = \{x\}$ for some $x \in \Omega$. Given a block Λ of G , the set $\Gamma := \{\Lambda^\alpha : \alpha \in G\}$ defines a partition of Ω into blocks. This set is called *a system of blocks containing Λ* and the cardinality of Λ divides the cardinality of Ω . G acts naturally on Γ . A transitive permutation group is *primitive* if it determines only trivial blocks. Otherwise it is *imprimitive*.

Proposition 1.1 (Corollary 1.5A, [7]). *Let G be a transitive permutation group on a set Ω with at least two points. Then G is primitive if and only if each isotropy subgroup G_x , for $x \in \Omega$, is a maximal subgroup of G .* \square

It will be important for us to recognize when a permutation α already provides primitivity for any subgroup that contains α .

Example 1.2. *If $\gcd(l, d) = 1$ and l is greater than any non-trivial divisor of d then any transitive permutation group $G < \Sigma_d$ containing a l -cycle is primitive (this holds, for example, if $d = 2l \pm 1$). In fact, we can assume that G contains the cycle $(1 \dots l)$. If there is a block of G containing two elements i and j with $i \leq l$ and $j > l$ then it also contains $1, \dots, l$, thus the cardinality of the block is $\geq l + 1$, hence it equals d and the block is trivial. Otherwise the cardinality of each block divides both l and $d - l$, hence it equals 1, thus all blocks of G are trivial. Hence G is a primitive permutation group.*

1.2 Branched coverings over the projective plane

A surjective continuous open map $\phi : M \longrightarrow N$ between closed surfaces such that:

- for $x \in N$, $\phi^{-1}(x)$ is a finite set, and
- there is a discrete set $B_\phi \subset N$ such that the restriction $\hat{\phi} := \phi|_{M \setminus \phi^{-1}(B_\phi)}$ is an ordinary unbranched covering of degree d ,

is called a *branched covering of degree d over N* and it is denoted by (M, ϕ, N, B_ϕ, d) . The surface N is the *base surface*, M is the *covering surface* and B_ϕ is the *branch point set*. If $B_\phi = \emptyset$ then we also call ϕ an *unbranched covering*, and if $B_\phi \neq \emptyset$ then we also call ϕ a *proper branched covering*. Its *associated unbranched covering* is denoted by $(\widehat{M}, \hat{\phi}, \widehat{N}, d)$, where $\widehat{N} := N \setminus B_\phi$ and $\widehat{M} := M \setminus \phi^{-1}(B_\phi)$. It is known that $\chi(\widehat{M}) = d\chi(\widehat{N})$, equivalently

$$\chi(M) - \#\phi^{-1}(B_\phi) = d(\chi(N) - \#B_\phi). \quad (1)$$

The set B_ϕ contains the image of the set of the points in M in that ϕ fails to be a local homeomorphism. Then each $x \in B_\phi$ determines a partition D_x of d (possibly $D_x = [1, \dots, 1]$), defined by the local degrees of ϕ on each component in the preimage of a small disk U_x around x , with $U_x \cap B_\phi = \{x\}$. The collection $\mathcal{D} := \{D_x\}_{x \in B_\phi}$ is called the *branch datum* and its *total defect* is the non-negative integer defined by $\nu(\mathcal{D}) := \sum_{x \in B_\phi} \nu(D_x)$. The total defect satisfies the *Riemann-Hurwitz formula* (see [10] or [8]):

$$\nu(\mathcal{D}) = d\chi(N) - \chi(M) \equiv 0 \pmod{2}. \quad (2)$$

Associated to (M, ϕ, N, B_ϕ, d) we have a permutation group, *the monodromy group of ϕ* , denoted by $G(\phi)$, which is the image of the *Hurwitz representation*

$$\rho_\phi : \pi_1(N \setminus B_\phi, z) \longrightarrow \Sigma_d, \quad (3)$$

that sends each class $\alpha \in \pi_1(N \setminus B_\phi, z)$ to a permutation of $\phi^{-1}(z) = \{z_1, \dots, z_d\}$, which indicates the terminal point of the lifting of a loop in α after fixing the initial point [10]. In particular, for $x \in B_\phi$, let c_x be a path from z to a small circle a_x about x and define the loop class $\mathbf{u}_x := [c_x a_x c_x^{-1}]$. Then the cyclic structure (see §1.1) of the permutation $\alpha_x := \rho_\phi(\mathbf{u}_x)$ is given by D_x and $\nu(\prod_{x \in B_\phi} \alpha_x) \equiv \nu(\mathcal{D}) \pmod{2}$.

Theorem 1.3 (See [11] and [9]). *Let N be a closed connected surface, $\mathcal{D} = \{D_1, \dots, D_s\}$ a finite collection of partitions of d and $F = \{x_1, \dots, x_s\} \subset N$ such that $\#F = s = \#\mathcal{D}$. If it is possible to define a representation $\rho : \pi_1(N \setminus F, z) \longrightarrow \Sigma_d$ with $\rho(\mathbf{u}_{x_i}) \in D_i$, $1 \leq i \leq s$ (and with a transitive image $\text{Imp} \rho = \rho(\pi_1(N \setminus F, z)) < \Sigma_d$), then \mathcal{D} is realizable on N , i.e. it is the branch datum of a branched covering on N (resp. with a connected covering surface M) having ρ as its Hurwitz representation. \square*

Remark 1.4. *If $N = \mathbb{R}P^2$ and $\mathcal{D} = \{D_1, \dots, D_s\}$, in order to define ρ_ϕ , we need at least permutations $\alpha_i \in D_i$, for $i = 1, \dots, s$, such that $\prod_{i=1}^s \alpha_i$ is a square, as result of the presentation of $\pi_1(\mathbb{R}P^2 \setminus \{x_1, \dots, x_s\}) = \langle a, \mathbf{u}_1, \dots, \mathbf{u}_s \mid \prod_{i=1}^s \mathbf{u}_i = a^{-2} \rangle$. We will also need the transitivity of the subgroup generated by these permutations, in order to obtain a connected covering surface M .*

Example 1.5. *If $r > 0$ is an odd natural number then every r -cycle is the square of a permutation: if $\alpha = (a_1 a_2 \dots a_r)$ then $\alpha = \beta^2$ where*

$$\beta = (a_1 a_{\frac{r+1}{2}+1} a_2 a_{\frac{r+1}{2}+2} \dots a_r a_{\frac{r+1}{2}}).$$

We state below the main theorem from [8] about the realizability of branched covering over $\mathbb{R}P^2$ in a slightly different form, which is more suitable for our purpose.

Theorem 1.6 (See [8]). *Let \mathcal{D} be a collection of partitions of d . Then there is a branched covering $\phi : M \rightarrow \mathbb{R}P^2$ of degree d , with M connected and nonorientable and with branch datum \mathcal{D} if and only if*

$$d - 1 \leq \nu(\mathcal{D}) \equiv 0 \pmod{2}. \quad (4)$$

Moreover, if $\phi : M \rightarrow \mathbb{R}P^2$ is a branched covering and $\phi_\# : \pi_1(M) \rightarrow \pi_1(\mathbb{R}P^2)$ is trivial then M is orientable. If $\phi : M \rightarrow \mathbb{R}P^2$ is a branched covering and $\phi_\# : \pi_1(M) \rightarrow \pi_1(\mathbb{R}P^2)$ is surjective then M is nonorientable. \square

In the cases where the covering space is orientable, the problem does not appear as a problem over $\mathbb{R}P^2$ but will lead naturally to a similar question for a branched covering over S^2 .

Finally we conclude this section by observing that from the *Riemann-Hurwitz formula* above we have $\nu(\mathcal{D}) = d - 1$ if and only if $\chi(M) = 1$ which is equivalent to say that M is homeomorphic to $\mathbb{R}P^2$. Also if $\nu(\mathcal{D}) = d$ by the Hurwitz condition (2) follows that d is even. Therefore if d is odd we can not have $\nu(\mathcal{D}) = d$.

2 Decomposability

Given a branched covering, it is *decomposable* if it can be written as a composition of two non-trivial branched coverings (i.e. both with degree bigger than 1), otherwise it is called *indecomposable*. In a decomposition of a proper branched covering, at least one of its factors is a proper branched covering. Moreover, since the degree of a decomposable covering is the product of the degrees of its factors (see [4], theorem 2.3), we are interested in branched coverings with non-prime degree.

Proposition 2.1 (Proposition 2.8 [2]). *The covering surface M of a branched covering is connected if and only if its monodromy group is transitive. A branched covering with a connected covering surface is decomposable if and only if its monodromy group is imprimitive.* \square

Proposition 2.2. *A branched covering $(\mathbb{R}P^2, \phi, \mathbb{R}P^2, \{x\}, d)$ is decomposable if and only if d is not a prime.*

Proof. From (1) and (2) the total defect of the branch datum is $d - 1$ and by (4), d is odd. So the branched covering $(\mathbb{R}P^2, \phi, \mathbb{R}P^2, \{x\}, d)$ has branch datum $\mathcal{D} = \{[d]\}$. In a representation

$$\begin{aligned} \rho : \pi_1(\mathbb{R}P^2 \setminus \{x\}) = \langle a, \mathbf{u}_x | a^2 \mathbf{u}_x = 1 \rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha, \\ \mathbf{u}_x &\longmapsto \gamma, \end{aligned}$$

where $\alpha^2 = \gamma^{-1}$ is a d -cycle, necessarily α is a d -cycle. Therefore, in the group $G := \text{Im} \rho = \langle \gamma, \alpha \rangle = \langle \alpha \rangle$, every isotropy subgroup is trivial and, if $d > 1$ is not a prime, it is contained in a proper subgroup of G . Then by Proposition 1.1, G is imprimitive, and by Proposition 2.1, the branched covering is decomposable.

The inverse implication is obvious. \square

Lemma 2.3. *Let $\alpha \in \Sigma_d$ be an even permutation such that $\nu(\alpha) < d - 1$ and either $\text{Fix}(\alpha) \neq \emptyset$ or $\alpha^2 \neq 1_d$. Then there exists a d -cycle $\beta \in \Sigma_d$ such that $\alpha\beta$ is also a d -cycle and $H = \langle \alpha, \beta \rangle$ is a primitive permutation group.*

Proof. Case (3) of Theorem 3.2 in [2]. □

Proposition 2.4. *Let $d > 1$ be odd and $\mathcal{D} = \{D_1, \dots, D_s\}$ a nonorientable-admissible datum of degree d (see (4) or Theorem 1.6). If there is $i \in \{1, \dots, s\}$ such that $D_i = [d]$ and $D_j \neq [1, \dots, 1]$ for all $j \neq i$, then \mathcal{D} is realizable by an indecomposable (and hence primitive) branched covering over $\mathbb{R}P^2$ if and only if d is prime or $s > 1$.*

Proof. Suppose $\mathcal{D} = \{D_1, \dots, D_s\}$, $s > 1$ and without loss of generality suppose $D_s = [d]$. For $i = 1, \dots, s-1$ choose $\gamma_i \in \Sigma_d$ with cyclic structure D_i .

If $\prod_{i=1}^{s-1} \gamma_i \neq 1_d$, then its cyclic structure determines a new partition $D = [d_1, \dots, d_q]$ of d such that $\nu(D) = d - q \equiv \sum_{i=1}^{s-1} \nu(D_i) = \nu(\mathcal{D}) - \nu(D_s) \equiv d - 1 \equiv 0 \pmod{2}$, then q is odd. If $q = 1$, define $\gamma_s := (\prod_{i=1}^{s-1} \gamma_i)^{-1}$, $\alpha := (1 \ 1^{\gamma_s})$ and the representation

$$\begin{aligned} \rho : \langle a, \{\mathbf{u}_i\}_{i=1}^s \mid a^2 \prod_{i=1}^s \mathbf{u}_i = 1 \rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha, \\ \mathbf{u}_i &\longmapsto \gamma_i. \end{aligned}$$

Notice that $\text{Imp} \rho$ is a primitive permutation group, because by the structure of α , every block containing the element 1 contains also $1^\alpha = 1^{\gamma_s}$ and since γ_s is a d -cycle, this block contains everything, therefore the block is trivial.

If $q > 1$, we apply Lemma 2.3 for $\prod_{i=1}^{s-1} \gamma_i$ and therefore there is a d -cycle γ_s such that $\prod_{i=1}^s \gamma_i$ is a d -cycle and $\langle \prod_{i=1}^{s-1} \gamma_i, \gamma_s \rangle$ is primitive. Moreover, since $\prod_{i=1}^s \gamma_i$ is an odd length cycle, by Example 1.5 there is $\alpha \in \Sigma_d$ such that $\alpha^2 = \prod_{i=1}^s \gamma_i$. We define the following representation:

$$\begin{aligned} \rho : \langle a, \{\mathbf{u}_i\}_{i=1}^s \mid a^2 \prod_{i=1}^s \mathbf{u}_i = 1 \rangle &\longrightarrow \Sigma_d \\ a &\longmapsto \alpha^{-1}, \\ \mathbf{u}_i &\longmapsto \gamma_i, \end{aligned}$$

with $\text{Imp} \rho$ primitive because it contains $\langle \prod_{i=1}^{s-1} \gamma_i, \gamma_s \rangle$. Since $\text{Imp} \rho$ is primitive in both cases above, Proposition 2.1 guarantees that branched coverings associated by virtue of Theorem 1.3 to each one of the representations above are indecomposable.

If $\prod_{i=1}^{s-1} \gamma_i = 1_d$ and there is some γ_i with a cycle with length ≥ 3 , we change γ_i by γ_i^{-1} . If each γ_i is a product of independent cycles of length less than or equal to 2 we replace the symbol of a transposition (which exists because $D_j \neq [1, \dots, 1]$ for some $j = 1, \dots, s-1$) by a symbol in another cycle. Thus, without changing the cyclic structure of the permutations, the new product $\prod_{i=1}^s \gamma_i$ is different from the identity and we are in the case before.

The implications follow immediately from Proposition 2.2 and Theorem 1.6. \square

Remark 2.5. *Observe that, among the nonorientable-admissible data studied in the previous proposition, the ones that are realized by indecomposable branched coverings over $\mathbb{R}P^2$ are such that $\nu(\mathcal{D}) > d - 1$ or d is prime. We want to know if this property is enough to guarantee the existence of an indecomposable realization over $\mathbb{R}P^2$ for any nonorientable-admissible branch data of odd degree d . For that it remains to analyze the cases where the partitions in \mathcal{D} are all different from $[d]$ and d is odd and non-prime.*

3 The case of 2 branch points

In this section we study the problem in the special case where the number of branch points is two (which is the minimal possible value, provided that $\nu(\mathcal{D}) \geq d$). In the next section we will show that the general case can be reduced to this case. The main result of the section is:

Theorem 3.1. *If $\mathcal{D} = \{D_1, D_2\}$ is such that d is odd and $d < \nu(\mathcal{D}) \equiv 0 \pmod{2}$ then \mathcal{D} is realizable by an indecomposable branched covering over $\mathbb{R}P^2$ with a connected covering surface.*

To show the above result the main tool is the Lemma 3.4, which can be useful for other applications. Several of the arguments used in the proof of this Lemma are similar to some arguments which appear in the proof of the lemma below:

Lemma 3.2 (Corollary 4.4 [8]). *Let $\overline{D}_1, \overline{D}_2$ be partitions of $\overline{d} \in \mathbb{N}$ such that $\nu(\overline{D}_1) + \nu(\overline{D}_2) \geq \overline{d} - 1$ and $\nu(\overline{D}_1) + \nu(\overline{D}_2) \equiv \overline{d} + 1 \pmod{2}$. Given $\overline{\lambda} \in \overline{D}_1$ there exists $\overline{\beta} \in \overline{D}_2$ such that $\overline{\lambda}\overline{\beta}$ is a \overline{d} -cycle (and hence $\langle \overline{\lambda}, \overline{\beta} \rangle$ acts transitively on $\{1, \dots, \overline{d}\}$). \square*

Although the proof of Lemma 3.4 is more elaborate. The above Lemma is not sufficient to get our Lemma.

It is worth to say that due to Proposition 2.4 and Remark 2.5, $\mathcal{D} = \{D_1, D_2\}$ can be assumed to be a nonorientable-admissible datum such that $d > 1$ is odd and non-prime, moreover $\nu(D_i) < d - 1$ and, hence $d - 1 < \nu(\mathcal{D}) < 2(d - 1)$. Since $\nu(\mathcal{D})$ is even and d is odd then $\nu(\mathcal{D}) > d$. Moreover, since d is a non-prime odd integer then $d \geq 9$. But the case $d = 9$ is completely solved in [3]. Because of this, it suffices to study the case $d > 9$. Nevertheless we give a complete proof for all $d \geq 3$ based on Lemma 3.4, which is interesting in its own right.

Notation. With notation and conditions of Lemma 3.2, we will write

$$\bar{\beta} := EKS(\bar{d}, \bar{\lambda}, \bar{D}_2)$$

to mean: $\bar{\beta}$ is the permutation obtained by applying Lemma 3.2. The notation “EKS” comes from the first letters of the names of the authors of [8].

Remark 3.3. Let Ω be a set with d elements and suppose $\bar{\Omega} \subset \Omega$ be a proper subset with \bar{d} elements, i.e. $\bar{d} < d$. Notice that there exists a projection (not a homomorphism):

$$\begin{array}{ccc} \wp : \text{Sym}_{\Omega} & \rightarrow & \text{Sym}_{\bar{\Omega}} \\ \lambda & \mapsto & \wp(\lambda) \end{array} \quad (5)$$

where $\wp(\lambda)$ is obtained from λ by deleting the elements of $\Omega \setminus \bar{\Omega}$ in the cyclic decomposition of λ (see beginning of subsection 1.1). There exists also a monomorphism:

$$\begin{array}{ccc} \iota : \text{Sym}_{\bar{\Omega}} & \rightarrow & \text{Sym}_{\Omega} \\ \bar{\beta} & \mapsto & \iota(\bar{\beta}) \end{array} \quad (6)$$

such that

$$[\iota(\bar{\beta})](i) := \begin{cases} i, & \text{if } i \notin \bar{\Omega}, \\ \bar{\beta}(i), & \text{if } i \in \bar{\Omega}. \end{cases} \quad (7)$$

Consider $\lambda \in \text{Sym}_{\Omega}$ and $\bar{\beta} \in \text{Sym}_{\bar{\Omega}}$. We want to understand how to obtain $\lambda\iota(\bar{\beta})$ from $\wp(\lambda)\bar{\beta}$ and λ . Suppose $\bar{\Omega} := \{\mathbf{w}_1, \dots, \mathbf{w}_{\bar{d}}\}$ and

$$\lambda = (\mathbf{w}_1 \underbrace{u_{1,1} \dots u_{1,t_1}}_{S_1} \mathbf{w}_2 \underbrace{u_{2,1} \dots u_{2,t_2}}_{S_2} \dots \mathbf{w}_{\bar{d}} \underbrace{u_{\bar{d},1} \dots u_{\bar{d},t_{\bar{d}}}}_{S_{\bar{d}}})$$

where for $i = 1, \dots, \bar{d}$ we have $u_{i,j} \in \Omega \setminus \bar{\Omega}$ for $j = 1, \dots, t_i$ and $t_i \geq 0$, i.e. the sequences $S_i = u_{i,1} \dots u_{i,t_i}$ can be empty and $\bar{d} + \sum_{i=1}^{\bar{d}} t_i \leq d$. Then

$$\wp(\lambda) = (\mathbf{w}_1 \mathbf{w}_2 \dots \mathbf{w}_{\bar{d}}).$$

Suppose also that

$$\wp(\lambda)\bar{\beta} = (\mathbf{w}_{\phi(1)} \dots \mathbf{w}_{\phi(v_1)}) (\mathbf{w}_{\phi(v_1+1)} \dots \mathbf{w}_{\phi(v_1+v_2)}) \dots (\mathbf{w}_{\phi(v+1)} \dots \mathbf{w}_{\phi(v+v_x)})$$

where $v := v_1 + v_2 + \dots + v_{x-1}$, $v + v_x = \bar{d}$ and $\phi \in \text{Sym}_{\bar{d}}$. Then

$$[\lambda\iota(\bar{\beta})](q) = [\iota(\bar{\beta})](\lambda(q)) = \begin{cases} \bar{\beta}(\lambda(q)), & \text{if } \lambda(q) \in \bar{\Omega}, \\ \lambda(q), & \text{if } \lambda(q) \notin \bar{\Omega}, \end{cases} \quad (8)$$

but for $\lambda(q) \in \overline{\Omega}$ we have

$$\begin{aligned} \bar{\beta}(\lambda(q)) &= [\wp(\lambda)]^{-1}[\wp(\lambda)\bar{\beta}](\lambda(q)) = \\ &= [\wp(\lambda)\bar{\beta}](\lambda(q)) = \begin{cases} [\wp(\lambda)\bar{\beta}](q), & \text{if } q \in \overline{\Omega}, \\ [\wp(\lambda)\bar{\beta}](\mathbf{w}_l), & \text{if } q \notin \overline{\Omega} \text{ and } q \in S_l. \end{cases} \end{aligned} \quad (9)$$

Hence

$$\begin{aligned} \lambda_i(\bar{\beta}) &= (\mathbf{w}_{\phi(1)} S_{\phi(1)} \dots \mathbf{w}_{\phi(v_1)} S_{\phi(v_1)}) (\mathbf{w}_{\phi(v_1+1)} S_{\phi(v_1+1)} \dots \mathbf{w}_{\phi(v_1+v_2)} S_{\phi(v_1+v_2)}) \dots \\ &\dots (\mathbf{w}_{\phi(v+1)} S_{\phi(v+1)} \dots \mathbf{w}_{\phi(v+v_x)} S_{\phi(v+v_x)}). \end{aligned}$$

Thus we obtain the cyclic decomposition of $\lambda_i(\bar{\beta})$ from the cyclic decomposition of $\wp(\lambda)\bar{\beta}$ by replacing each \mathbf{w}_i (determined via λ) by $\mathbf{w}_i S_i$, for $i = 1, \dots, \bar{d}$.

If λ is a product of several disjoint cycles, we use the procedure from above on each cycle to obtain $\lambda_i(\bar{\beta})$. Namely, we first define the sequence S_i which is the consecutive of \mathbf{w}_i , using the procedure above to the cycle which contains \mathbf{w}_i . Then define $\lambda_i(\bar{\beta})$ as above. Moreover, if λ contains cycles whose elements are totally contained in $\overline{\Omega}$, by (8) and (9) the corresponding cycles of $\wp(\lambda)\bar{\beta}$ are cycles of $\lambda_i(\bar{\beta})$.

Lemma 3.4. Let $\mathcal{D} = \{D_1, D_2\}$ be a pair of partitions of an odd integer $d \geq 3$ such that $d - 1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$. Given a permutation $\lambda \in D_1$, there exists $\beta \in D_2$ such that $\lambda\beta$ is a $(d-2)$ -cycle and the permutation group $G := \langle \lambda, \beta \rangle \leq \Sigma_d$ is transitive. For any such a collection of permutations, the permutation group G is primitive.

Proof. In the case of $d = 3$, we have $D_1 = D_2 = [3]$, and $\beta = \lambda^{-1}$ has the desired properties. From now on, we will assume that $d > 3$.

Let $\nu(\mathcal{D}) = (d - 1) + r$, with $r > 0$ even. Due to Proposition 2.4, we can suppose

$$D_1 = [c_1, c_2, \dots, c_t], \quad D_2 = [d_1, d_2, \dots, d_s, \underbrace{1, \dots, 1}_{\ell}], \quad \text{with} \quad (10)$$

$$1 < t < d, \quad 1 < s + \ell < d, \quad d - 1 > \nu(D_1) \geq \nu(D_2), \quad \text{and} \quad (11)$$

$$c_1 \geq c_2 \geq \dots \geq c_t, \quad d_1 \geq d_2 \geq \dots \geq d_s > 1. \quad (12)$$

Let $\Omega = \{1, \dots, d\}$ and $\lambda \in \text{Sym}_{\Omega}$ be a permutation with the cyclic decomposition $\lambda = \lambda_1 \dots \lambda_t \in D_1$, where $\lambda_i := (a_{i,1} a_{i,2} \dots a_{i,c_i})$ is a c_i -cycle, for $i = 1, \dots, t$.

Since $\nu(D_1) + \nu(D_2) \geq d + 1$ with $\nu(D_1) \geq \nu(D_2)$, we have

$$\nu(D_1) \geq (d + 1)/2, \quad \text{and } c_1 \geq 3. \quad (13)$$

Here is the plan of the proof. In order to construct $\beta \in D_2$, we will divide the problem into three cases. In each case we want to define the first cycle β_1 in a cyclic decomposition $\beta = \beta_1 \dots \beta_s$ of β , in order to guarantee that $|\text{Fix}(\lambda\beta)| \geq 2$ (i.e. there exist at least two fixed elements of the permutation $\lambda\beta$). Then we want to study a related problem in the symmetric group of order $d - 2$. Namely, we need a stronger version of the Lemma 3.2.

If we are able to make the construction in such a way that the first cycles $\bar{\lambda}_1$ and $\bar{\beta}_1$ in a solution of this related problem have supports with non-empty intersection, then we are able to solve the original problem in the symmetric group of order d .

First case: $c_1 + d_1 > 6$ and $d_1 \geq 3$.

Step 1. Define

$$\beta_0 := (a_{1,3} \ a_{1,2} \ a_{1,1}), \text{ then } \lambda\beta_0 = (\overline{a_{1,1}})(\overline{a_{1,2}})(a_{1,3} \ \dots \ a_{1,c_1})\lambda_2 \dots \lambda_t,$$

where $\overline{a_{i,j}}$, means “used elements” in the sense that they will be elements in the support of the d_1 -cycle β_1 of β and we cannot use them to define other cycles of β . Let $\bar{d} := d - 2$ and let

$$\bar{D}_1 := [c_1 - 2, c_2, \dots, c_t], \quad \bar{D}_2 := [d_1 - 2, d_2, \dots, d_s, \underbrace{1, \dots, 1}_\ell]$$

be partitions of \bar{d} . Then

$$\nu(\bar{D}_1) + \nu(\bar{D}_2) = \nu(\mathcal{D}) - 4 = (\bar{d} - 1) + (r - 2) \equiv \bar{d} + 1 \pmod{2}. \quad (14)$$

Put $\bar{\Omega} := \Omega \setminus \{a_{1,1}, a_{1,2}\}$. Then

$$\wp(\lambda\beta_0) = \wp(\lambda) = \wp(\lambda_1)\lambda_2 \dots \lambda_t \quad (15)$$

is a cyclic decomposition of $\wp(\lambda)$.

Step 2. On this step, we will work in the permutation group $\text{Sym}_{\bar{\Omega}}$. We will prove that the permutation $\bar{\beta} \in \bar{D}_2$ from Lemma 3.2 can be chosen in such a way that $a_{1,3}$ belongs to the support of a $(d_1 - 2)$ -cycle of a cyclic decomposition of $\bar{\beta}$. Consider two subcases.

Subcase 1: $d_1 \geq 3$ and $r = 2$. In this subcase, $\nu(\bar{D}_1) + \nu(\bar{D}_2) = \bar{d} - 1$ and $t = \nu(\bar{D}_2) + 1$, hence

$$t = (d_1 - 2) + (d_2 - 1) + \dots + (d_s - 1)$$

and this suggests how to define $\bar{\beta} \in \bar{D}_2$ such that $\wp(\lambda)\bar{\beta}$ will be a \bar{d} -cycle. First we define the $(d_1 - 2)$ -cycle $\bar{\beta}_1$ of $\bar{\beta}$ by using for its support the element $a_{1,3}$ and

one element in $\text{Supp}(\lambda_i)$ for $i = 2, \dots, d_1 - 2$, if $d_1 - 1 > 3$ or $\bar{\beta}_1 = (a_{1,3})$ if $d_1 = 3$. So in the product $\wp(\lambda)\bar{\beta}_1$, the elements of the supports of the first $(d_1 - 2)$ cycles in the cyclic decomposition (15) form the support of a $((\sum_{j=1}^{d_1-2} c_j) - 2)$ -cycle Λ_1 . In other words, $\wp(\lambda)\bar{\beta}_1$ will have the cyclic decomposition of the form

$$\wp(\lambda)\bar{\beta}_1 = \Lambda_1 \lambda_{d_1-1} \dots \lambda_t,$$

with the number of cycles being $d_2 + (d_3 - 1) \dots + (d_s - 1)$, and $\text{Supp}(\Lambda_1) = \text{Supp}(\wp(\lambda_1)) \cup \bigcup_{k=2}^{d_1-2} \text{Supp}(\lambda_k)$.

Suppose that $1 \leq i \leq s - 1$. Let $k_i := (d_1 - 2) + \sum_{j=2}^i (d_j - 1)$ and $m_i := (c_1 - 2) + \sum_{l=2}^{k_i} c_l$. Suppose that we have constructed a $(d_1 - 2)$ -cycle $\bar{\beta}_1$ and d_j -cycles $\bar{\beta}_j$ with $2 \leq j \leq i$ having the following properties: a cyclic decomposition (see subsection 1.1) of the permutation $\wp(\lambda)\bar{\beta}_1 \dots \bar{\beta}_j$ has the form

$$\wp(\lambda)\bar{\beta}_1 \dots \bar{\beta}_j = \Lambda_j \lambda_{k_j+1} \dots \lambda_{k_{j+1}} \dots \lambda_t, \quad 1 \leq j \leq i, \quad (16)$$

and

$$\text{Supp}(\bar{\beta}_j) \cap \text{Supp}(\bar{\beta}_k) = \emptyset, \quad 1 \leq j < k \leq i, \quad (17)$$

where Λ_j is a m_j -cycle with $\text{Supp}(\Lambda_j) = \text{Supp}(\wp(\lambda_1)) \cup \bigcup_{k=2}^{k_j} \text{Supp}(\lambda_k)$. We want to determine a d_{i+1} -cycle $\bar{\beta}_{i+1}$ with the similar properties. By using the procedure of the constructing $\bar{\beta}_1$, we construct $\bar{\beta}_{i+1}$ such that its support contains one element from the support of each of the cycles $\Lambda_i, \lambda_{k_i+1}, \dots, \lambda_{k_{i+1}}$, and does not intersect the supports of any of $\bar{\beta}_1, \dots, \bar{\beta}_i$. In order to be able to make this construction for each $i = 1, \dots, s - 1$, it is sufficient to show the following inequality:

$$|\text{Supp}(\Lambda_i) \setminus \text{Supp}(\bar{\beta}_1 \dots \bar{\beta}_i)| \geq 1, \quad (18)$$

where

$$|\text{Supp}(\Lambda_i) \setminus \text{Supp}(\bar{\beta}_1 \dots \bar{\beta}_i)| = m_i - (d_1 - 2 + d_2 + \dots + d_i) = m_i - (k_i + i - 1)$$

$$= c_1 - 2 + \left(\sum_{l=2}^{k_i} c_l \right) - i + 1 = ((c_1 - 3) + \sum_{l=2}^{k_i} (c_l - 1)) - i + 1.$$

To prove the inequality (18), let n be the number of all members of the partition \bar{D}_1 , which are greater than 1. We have two cases:

1. if $n \geq k_i$ then $((c_1 - 3) + \sum_{l=2}^{k_i} (c_l - 1)) - i + 1 =$

$$\begin{aligned} & (c_1 - 3) + \sum_{l=2}^i (c_l - 2) + \sum_{l=i+1}^{k_i} (c_l - 1) \geq \\ & \geq 0 + 0 + 1 = 1, \end{aligned}$$

2. if $n < k_i$ then $\nu(\overline{D}_1) = c_1 - 3 + \sum_{l=2}^n (c_l - 1) = c_1 - 3 + \sum_{l=2}^{k_i} (c_l - 1)$, thus

$$\begin{aligned} ((c_1 - 3) + \sum_{l=2}^{k_i} (c_l - 1)) - i + 1 &= \\ \nu(\overline{D}_1) - i + 1 &= \\ (s + \ell) - i &\geq (s + \ell) - (s - 1) = \ell + 1 \geq 1, \end{aligned}$$

so the inequality (18) holds.

Define $\bar{\beta} := \bar{\beta}_1 \dots \bar{\beta}_s$, then $k_s = t$ and $|\text{Supp}(\Lambda_s)| = m_s = (c_1 - 2) + \sum_{l=2}^t c_l = d - 2 = \bar{d}$, thus

$$\wp(\lambda)\bar{\beta} = \Lambda_s \quad (19)$$

is a \bar{d} -cycle by (16) for $j = i = s$, and a cyclic decomposition of $\bar{\beta}$ is $\bar{\beta} = \bar{\beta}_1 \dots \bar{\beta}_s$ by (17) for $i = s$.

Remark 3.5. If $S \subset \text{Fix}(\bar{\beta})$,

$$\Lambda_s = (u_1 v_1 \dots w_1 \ u_2 v_2 \dots w_2 \ \dots \ \dots \ \dots \ u_z v_z \dots w_z \ * v_* \dots w_*),$$

where $\{u_1, \dots, u_z\} = S$. This is going to be useful in Subsubcase 2b.

Subcase 2: $d_1 \geq 3$ and $r > 2$.

We start by reordering, in an increasing way, the entries of

$$\overline{D}_2 = [d_1 - 2, d_2, \dots, d_s, 1, \dots, 1],$$

i.e. put

$$\overline{D}_2 = [e_1, \dots, e_\ell, e_{\ell+1}, \dots, e_{\ell+s}]$$

with $1 = e_1 = \dots = e_\ell < e_{\ell+1} \leq e_{\ell+2} \leq \dots \leq e_{\ell+s}$. Since $r > 2$ and $\nu(\overline{D}_1) + \nu(\overline{D}_2) = (\bar{d} - 1) + (r - 2)$, there is $0 \leq k < s$ such that

$$\nu(\wp(\lambda)) + (e_{\ell+1} - 1) + \dots + (e_{\ell+k} - 1) \leq \bar{d} - 1$$

and

$$\nu(\wp(\lambda)) + (e_{\ell+1} - 1) + \dots + (e_{\ell+k} - 1) + (e_{\ell+k+1} - 1) > \bar{d} - 1.$$

Let $f > 0$ such that

$$\nu(\wp(\lambda)) + (e_{\ell+1} - 1) + \dots + (e_{\ell+k} - 1) + (f - 1) = \bar{d} - 1. \quad (20)$$

Define

$$\overline{D}_{2,1} := [e_1, \dots, e_\ell, e_{\ell+1}, \dots, e_{\ell+k}, f, \underbrace{1, \dots, 1}_z],$$

a partition of \bar{d} where $z := \bar{d} - (\sum_{i=1}^{\ell+k} e_i) - f$ is bigger than zero, and define

$$\bar{D}_{2,2}(j) := [e_{\ell+k+1}, \dots, e_{\ell+j-1}, e_{\ell+j} - f + 1, e_{\ell+j+1}, \dots, e_{\ell+s}],$$

a partition of $z + 1$, where $j \in \{k + 1, \dots, s\}$. Later, we will make a choice of j .

Here is the plan of constructing $\bar{\beta}$ in Subcase 2.

Firstly, we will define $\beta' \in \bar{D}_{2,1}$ and $\beta'' \in \bar{D}_{2,2}(j)$ such that $\text{Supp}(\beta') \cap \text{Supp}(\beta'')$ consists of only one element denoted by $*$ in $\bar{\Omega}$. Secondly, we will put $\bar{\beta} := \beta' \iota_2(\beta'')$, where ι_2 is a monomorphism from $\text{Sym}_{\underline{\Omega}}$ into $\text{Sym}_{\bar{\Omega}}$, where $\underline{\Omega}$ will be a set with $z + 1$ element. Then we will check that $\bar{\beta}$ has the desired properties.

Defining $\beta' \in \bar{D}_{2,1}$. By (20) we have $\nu(\bar{D}_1) + \nu(\bar{D}_{2,1}) = \bar{d} - 1$. Thus

$$t = \nu(\bar{D}_{2,1}) + 1. \quad (21)$$

By (21) we have

$$t = (e_{\ell+1}) + (e_{\ell+2} - 1) + \dots + (e_{\ell+k} - 1) + (f - 1), \quad (22)$$

thus the permutation $\wp(\lambda)$ and the partition $\bar{D}_{2,1}$ satisfy the hypothesis of Subcase 1. Consider two subsubcases of Subcase 2.

Subsubcase 2a: the integer $d_1 - 2$ is a member of the partition $\bar{D}_{2,1}$, more precisely $d_1 - 2 = e_{j_0}$ for some $j_0 \in \{1, \dots, \ell + k\}$. We define $\beta' \in \bar{D}_{2,1}$ to be the permutation $\bar{\beta}$ constructed in Subcase 1 via the permutation $\wp(\lambda)$, the partition $\bar{D}_{2,1}$ and its distinguished entry $d_1 - 2$.

Subsubcase 2b: the integer $d_1 - 2$ is not a member of the partition $\bar{D}_{2,1}$, i.e. $d_1 - 2 = e_{j_0}$ for some $j_0 > \ell + k$.

We define $\beta' \in \bar{D}_{2,1}$ to be the permutation constructed in Subcase 1 via the permutation $\wp(\lambda)$, the partition $\bar{D}_{2,1}$ and its distinguished entry f , i.e. $\beta' \in \bar{D}_{2,1}$ satisfies the assertions of Lemma 3.2 and if $f > 1$, $a_{1,3}$ belongs to the support of an f -cycle of a cyclic decomposition of β' , otherwise $a_{1,3} \in \text{Fix}(\beta')$.

In any of the subsubcases 2a and 2b, a cyclic decomposition of β' has the form $\beta' = \epsilon_1 \dots \epsilon_{\ell+k} \epsilon_f$ where ϵ_i is a e_i -cycle for $i = 1, \dots, \ell + k$ and ϵ_f is an f -cycle with $a_{1,3} \in \text{Supp}(\epsilon_f)$ if $f > 1$, otherwise $a_{1,3} \in \text{Fix}(\beta')$. Let

$$F := \{\epsilon_1, \dots, \epsilon_\ell\} \subset \bar{\Omega} \setminus \{a_{1,3}\}$$

be the set of ℓ elements corresponding to the 1-cycles $\epsilon_1, \dots, \epsilon_\ell$.

Remark that, in Subsubcase 2a, we have $a_{1,3} \in \text{Supp}(\epsilon_{j_0})$, and we can choose any j in order to define β'' . In Subsubcase 2b, we have $a_{1,3} \in \text{Supp}(\epsilon_f)$ if $f > 1$, otherwise $a_{1,3} \in \text{Fix}(\beta')$, and define $j := j_0 - \ell$ and $*$:= $a_{1,3}$.

Defining $\beta'' \in \bar{D}_{2,2}(j)$. Here, we will define $\beta'' \in \bar{D}_{2,2}(j)$ as a permutation of the set

$$\underline{\Omega} := \{*, u_1, \dots, u_z\}$$

of $z + 1$ elements formed by the disjoint union of the element $*$ and the set

$$\{u_1, \dots, u_z\} := \overline{\Omega} \setminus (F \cup \text{Supp}(\epsilon_1 \dots \epsilon_{\ell+k} \epsilon_f) \cup \{a_{1,3}\}).$$

Notice that $\nu(\overline{D}_2) = \nu(\overline{D}_{2,1}) + \nu(\overline{D}_{2,2}(j))$, so by (14) and (20) we have $\nu(\overline{D}_{2,2}(j)) = r - 2$.

In order to define β'' , consider

$$\tau \in \overline{D}_{2,2}(j) = [e_{\ell+k+1}, \dots, e_{\ell+j-1}, e_{\ell+j} - f + 1, e_{\ell+j+1}, \dots, e_{\ell+s}]$$

in $\text{Sym}_{\underline{\Omega}}$, a permutation such that $*$ is in its $(e_{\ell+j} - f + 1)$ -cycle. Since $r > 2$ is even, β'' will be a non-trivial even permutation and we can write β'' as a product of two $(z + 1)$ -cycles by Lemma 3.1 of [8]. Put $\tau = \sigma\gamma$, where σ and γ are $(z + 1)$ -cycles.

As observed in the end of Step 1, the \bar{d} -cycle $\wp(\lambda)\beta'$ has the form

$$(u_1 v_1 \dots w_1 \ u_2 v_2 \dots w_2 \ \dots \ \dots \ u_z v_z \dots w_z \ * \ v_* \dots w_*) \quad (23)$$

where $\{v_1, \dots, w_1, v_2, \dots, w_2, \dots, v_z, \dots, w_z, v_*, \dots, w_*\} = \overline{\Omega} \setminus \underline{\Omega}$. By Lemma 3.3 of [9] there exists $\eta \in \text{Sym}_{\underline{\Omega}}$ such that $\eta\sigma^{-1}\eta^{-1} = (u_1 \ u_2 \dots u_z \ *)$ and $\eta(*) = *$. Define

$$\beta'' := \eta\tau\eta^{-1} \in \overline{D}_{2,2}.$$

Defining $\bar{\beta} \in \overline{D}_2$. Put $\bar{\beta} := \beta' \iota_2(\beta'')$, where $\iota_2 : \text{Sym}_{\underline{\Omega}} \rightarrow \text{Sym}_{\overline{\Omega}}$ is the monomorphism (6).

Let us check that $\bar{\beta} \in \overline{D}_2$. Notice that β' fix $\underline{\Omega} \setminus \{*\}$, then the product $\beta' \iota_2(\beta'')$ contains the cycles of β'' except its $e_{\ell+j} - f + 1$ cycle. Similarly $\iota_2(\beta'')$ fix the elements which corresponds to the cycles ϵ_i for $i = 1, \dots, \ell + k$, then the product $\beta' \iota_2(\beta'')$ contains the cycles of β' except its f -cycle. Finally the product restricted to the union of the elements in the f -cycle of β' and the elements in the $(e_{\ell+j} - f + 1)$ -cycle of β'' gives the cycle of length $e_{\ell+j}$ of $\bar{\beta}$.

Let us check that $a_{1,3}$ belongs to the support of a $(d_1 - 2)$ -cycle of a cyclic decomposition of $\bar{\beta}$. In subsubcase 2a, we have $a_{1,3} \in \text{Supp}(\epsilon_{j_0})$, and we can choose any j in order to define β'' . In subsubcase 2b, we have $a_{1,3} \in \text{Supp}(\epsilon_f)$ if $f > 1$, otherwise $a_{1,3} \in \text{Fix}(\beta')$, and we define $j := j_0 - \ell$ and $*$:= $a_{1,3}$.

Let us check that $\wp(\lambda)\bar{\beta}$ is a \bar{d} -cycle. The product $\wp(\lambda)\bar{\beta} = (\wp(\lambda)\beta') \iota_2(\beta'')$ can be described as being obtained from $(u_1 \dots u_z *)\beta''_2$ by replacing each u_j and $*$ by the sequences $u_j v_j \dots w_j$ and $*v_* \dots w_*$ respectively, occurring in $\wp(\lambda)\bar{\beta}_1$, as explained in Remark 3.3. Thus $\wp(\lambda)\bar{\beta}$ is a \bar{d} -cycle.

Step 3. On this step, we show how to construct $\beta \in \text{Sym}_{\Omega}$ via $\bar{\beta} \in \text{Sym}_{\overline{\Omega}}$ constructed on Step 2. By Step 2, a cyclic decomposition of $\bar{\beta}$ has the form $\bar{\beta} = \bar{\beta}_1 \dots \bar{\beta}_s$, for a $(d_1 - 2)$ -cycle $\bar{\beta}_1$ and d_j -cycles $\bar{\beta}_j$ with $2 \leq j \leq s$, where $a_{1,3} \in \text{Supp}(\bar{\beta}_1)$ or $(a_{1,3})$ is the 1-cycle which corresponds to $\bar{\beta}_1$.

Put $\beta := \beta_0 \iota(\bar{\beta})$. Observe that β_0 is a 3-cycle and $\bar{\beta}_1$ is a $(d_1 - 2)$ -cycle, moreover if $d_1 > 3$ then $\{a_{1,3}\} = \text{Supp}(\beta_0) \cap \text{Supp}(\iota(\bar{\beta}_1))$. In any case, it follows that $\beta_1 := \beta_0 \iota(\bar{\beta}_1)$ is a d_1 -cycle with

$$a_{1,3} \in \text{Supp}(\beta_1).$$

Therefore the permutation $\beta = \beta_0 \iota(\bar{\beta}) = \beta_1 \iota(\bar{\beta}_2 \dots \bar{\beta}_s) \in D_2$, since the cycles $\bar{\beta}_1, \dots, \bar{\beta}_s$ are disjoint and $\text{Supp}(\beta_0) \cap \text{Supp}(\iota(\bar{\beta}_i)) = \emptyset$ for $2 \leq i \leq s$. Observe that $\lambda\beta_0 = \iota(\wp(\lambda))$, which follows via the inclusion $\text{Supp}(\beta_0) \subseteq \text{Supp}(\lambda_1)$. Therefore, in Sym_Ω , we have

$$\lambda\beta = (\lambda\beta_0)\iota(\bar{\beta}) = \iota(\wp(\lambda)\bar{\beta})$$

is a $(d - 2)$ -cycle by Step 2, as required.

Second case: $c_1 = d_1 = 3$

- If $t = 2$, since $c_1 = 3$ and it is the biggest summand of D_1 , then $d \leq 6$.

If $d = 5$, from the table in the Appendix, line 1, the result follows.

- If $t = 3$, we have $5 \leq d \leq 9$. If $d = 5$, then $D_1 = D_2 = [3, 1, 1]$. But in this case $\nu(\mathcal{D}) = d - 1$, a contradiction with one hypothesis on the total defect. For $d = 7$ and $d = 9$, the possibilities for D_1, D_2 , as well as the realization for these cases, are given in the table in the Appendix, lines 2, 3, 4 and 5, 6, 7, respectively.

- If $t \geq 4$ then we have either $D_1 = D_2 = [3, 3, 2, 1]$, or $D_1 = [3, 3, 2, 1]$ and $D_2 = [3, 2, 2, 2]$, or $d = 11 = \nu(\mathcal{D}) - 1$, or $c_i \geq 2$ for $i = 1, 2, 3, 4$.

For the first two cases, the result follows from the table in the Appendix, lines 8, 9 and 10.

Let $d = 11 = \nu(\mathcal{D}) - 1$. Since $\nu(D_1) \geq (d + 1)/2$ then $\nu(D_1) \geq 6$ and $t \leq 5$. Then t is either 4 or 5. The possibilities for D_1, D_2 , as well the realization for these cases, are given by the table in the Appendix, lines 11 to 19.

So from now on let us assume that $t \geq 4$ and $c_i \geq 2$ for $i = 1, 2, 3, 4$.

Define

$$\beta_0 := (a_{1,2} \ a_{1,1} \ a_*)(a_{2,2} \ a_{2,1} \ a_{3,1})$$

where a_* exists just if $d_2 = 3$ and in this case $a_* := a_{4,1}$. Then $\lambda\beta_0 =$

$$= \begin{cases} (\overline{a_{1,1}})(\overline{a_{1,2}} \ a_{1,3})(\overline{a_{2,1}})(\overline{a_{2,2}} \dots a_{2,c_2} \ \overline{a_{3,1}} \dots a_{3,c_3}) \ \lambda_4 \dots \lambda_t, & \text{if } d_2 = 2, \\ (\overline{a_{1,1}})(\overline{a_{1,2}} \ a_{1,3} \ \overline{a_{4,1}} \dots a_{4,c_4})(\overline{a_{2,1}})(\overline{a_{2,2}} \dots a_{2,c_2} \ \overline{a_{3,1}} \dots a_{3,c_3}) \ \lambda_5 \dots \lambda_t, & \text{if } d_2 = 3, \end{cases}$$

Notice that $d_2 > 1$, if not $\nu(D_2) = 2$ and $\nu(D_1) = d - 1$, a contradiction with the hypothesis over $\nu(\mathcal{D})$ in the present section.

Let $\bar{d} := d - (d_1 + d_2)$ and let

$$\overline{D_1} := \begin{cases} [(c_1 - 2), (c_2 - 2) + (c_3 - 1), c_4, \dots, c_t], & \text{if } d_2 = 2, \\ [(c_1 - 2) + (c_4 - 1), (c_2 - 2) + (c_3 - 1), c_5, \dots, c_t], & \text{if } d_2 = 3, \end{cases}$$

and

$$\overline{D_2} := [d_3, \dots, d_s, 1, \dots, 1]$$

be partitions of \bar{d} . Then $\nu(\overline{D_1}) + \nu(\overline{D_2}) = \nu(\mathcal{D}) - (d_1 + d_2 + 2) = (\bar{d} - 1) + (r - 2) \equiv \bar{d} + 1 \pmod{2}$. Put $\overline{\Omega} := \Omega - \{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, a_{3,1}, a_*\}$.

Consider in $\text{Sym}_{\overline{\Omega}}$:

$$\wp(\lambda\beta_0) = \begin{cases} (a_{1,3})(a_{2,3} \dots a_{2,c_2} a_{3,2} \dots a_{3,c_3}) \lambda_4 \dots \lambda_t, & \text{if } d_2 = 2, \\ (a_{1,3} a_{4,2} \dots a_{4,c_4})(a_{2,3} \dots a_{2,c_2} a_{3,2} \dots a_{3,c_3}) \lambda_5 \dots \lambda_t, & \text{if } d_2 = 3 \end{cases}$$

in $\overline{D_1}$ and $\bar{\beta} := EKS(\bar{d}, \wp(\lambda\beta_0), \overline{D_2})$. By the Lemma 3.2, $\wp(\lambda)\bar{\beta}$ is a \bar{d} -cycle.

Put $\beta := \beta_0 \iota(\bar{\beta})$, thus $\lambda\beta$ will be the $(d - 2)$ -cycle constructed from $\wp(\lambda\beta_0)\bar{\beta}$ as explained in Remark 3.3.

Third case: $d_1 = 2$. We have $s \geq 2$. If not $\nu(D_2) = 1$ and $\nu(D_2) > d$, impossible. If c_1 is either 3 or 4, we have $c_2 \geq 3$, if not $\nu(D_1) \leq (d + 1)/2$ and since $\nu(D_2) \leq (d - 1)/2$ then $\nu(\mathcal{D}) \leq d$, a contradiction with the hypothesis. Define

$$\beta_0 := (a_{1,1} a_{1,2})(a_* a_{\#}),$$

where

$$a_* = \begin{cases} a_{2,2}, & \text{if } c_1 \leq 4 \\ a_{1,3}, & \text{if } c_1 > 4 \end{cases} \quad \text{and} \quad a_{\#} = \begin{cases} a_{2,1}, & \text{if } c_1 \leq 4 \\ a_{1,4}, & \text{if } c_1 > 4 \end{cases}.$$

Then

$$\lambda\beta_0 = \begin{cases} (\overline{a_{1,1}})(\overline{a_{1,2}} a_{1,3} \dots a_{1,c_1})(\overline{a_{2,1}})(\overline{a_{2,2}} \dots a_{2,c_2}) \lambda_3 \dots \lambda_t, & \text{if } c_1 \leq 4, \\ (\overline{a_{1,1}})(\overline{a_{1,3}})(\overline{a_{1,2}} \overline{a_{1,4}} \dots a_{1,c_1}) \lambda_2 \dots \lambda_t, & \text{if } c_1 > 4. \end{cases}$$

Let $\bar{d} := d - 4$ and let

$$\overline{D_1} := \begin{cases} [c_1 - 2, c_2 - 2, c_3, \dots, c_t], & \text{if } c_1 \leq 4 \\ [c_1 - 4, c_2, \dots, c_t], & \text{if } c_1 > 4 \end{cases}, \quad \overline{D_2} := [d_3, \dots, d_s, 1, \dots, 1]$$

be partitions of \bar{d} . Then $\nu(\bar{D}_1) + \nu(\bar{D}_2) = \nu(\mathcal{D}) - 6 = (\bar{d} - 1) + (r - 2) \equiv \bar{d} + 1 \pmod{2}$. Put $\bar{\Omega} := \Omega - \{a_{1,1}, a_{1,2}, a_*, a_\#\}$.

Whatever is the case, observe that $\bar{\Omega}$ is the set of “non-used elements” and define $\bar{\lambda} \in \Sigma_{\bar{\Omega}}$ to be obtained from $\lambda\beta_0$ by removing the “used elements” from its cyclic decomposition (thus $\bar{\lambda} \in \bar{D}_1$), and $\bar{\beta} := EKS(\bar{d}, \bar{\lambda}, \bar{D}_2)$. Then $\bar{\lambda}\bar{\beta}$ is a \bar{d} -cycle. Put $\beta := \beta_0\bar{\beta}$, thus $\lambda\beta$ is the $(d-2)$ -cycle constructed from $\bar{\lambda}\bar{\beta}$ by inserting the maximal subsequence $\overline{a_{i,j}} \dots \overline{a_{k,\ell}}$ of “used elements” (in any non-trivial cycle of $\lambda\beta_0$), next to $a_{m,n} := (\lambda\beta_0)^{-1}(\overline{a_{i,j}})$ on the right.

It remains to check that $\beta := \beta_0\bar{\beta} \in D_2 = [d_1, d_2, d_3, \dots, d_s, 1, \dots, 1]$. In the cases 2 and 3, this follows from the fact that the permutations $\beta_0 \in [d_1, d_2, 1, \dots, 1]$ and $\bar{\beta} \in \bar{D}_2 = [d_3, \dots, d_s, 1, \dots, 1]$ have disjoint supports (consisting of “used” and “non-used elements” respectively).

Finally, by considerations in the beginning of the proof, the theorem is proved. \square

Proof of Theorem 3.1

Proof. Since $\nu(\mathcal{D}) > d - 1$, we have $d \geq 2$. Since d is odd, we have $d \geq 3$. Let $\mathcal{D} = \{D_1, D_2\}$. By Lemma 3.4, there exist permutations $\sigma_i \in D_i$ for $i = 1, 2$ such that $\sigma_1\sigma_2$ is a $(d-2)$ -cycle and the permutation group $G := \langle \sigma_1, \sigma_2 \rangle \leq \Sigma_d$ is transitive and primitive.

Since $d-2$ is odd and $\sigma_1\sigma_2$ is a $(d-2)$ -cycle, we can define $\alpha := \sqrt{\sigma_1\sigma_2}$, so $G = \langle \alpha, \sigma_1, \sigma_2 \rangle$ and $\sigma_1\sigma_2 = \alpha^2$. Now we apply the Hurwitz approach to this triple of permutations. By Theorem 1.3 and Remark 1.4 we can realize G as the monodromy group of a branched covering $(M, \phi, \mathbb{R}P^2, B_\phi, d)$ whose branch datum is \mathcal{D} . Moreover, since the monodromy group G is transitive and primitive, by Proposition 2.1 the covering surface M is connected and the branched covering ϕ is indecomposable. \square

4 Arbitrary number of branch points

This section is devoted to the proof of the main theorem. We will do that by induction on the number of partitions of the branch datum. For that, it is necessary to know how we can reduce the size of the branch datum without loose relevant information. This procedure is inspired by the proof of Theorem 5.1[8].

From lemma 4.2[8] and lemma 4.3 [8] we can read the following two assertions, which we will state not as strong as they were given in [8].

Lemma 4.1. [Lemma 4.2 [8]] Let A, B be partitions of d with $\nu(A) + \nu(B) = d - t, t \geq 1$. Then there exist permutations $\alpha \in A$ and $\beta \in B$ such that $\nu(\alpha\beta) = d - t$. \square

Lemma 4.2. [Lemma 4.3 [8]] Let A, B be partitions of d with $\nu(A) + \nu(B) = (d - 1) + r, r > 0$. Then for each k satisfying $0 \leq k \leq r, k \equiv r \pmod{2}$, one may choose $\alpha \in A, \beta \in B$ so that $\nu(\alpha\beta) = (d - 1) - k$. \square

Proposition 4.3. Let $\mathcal{D} = \{D_1, \dots, D_s\}$ be a collection of partitions of an odd integer $d \geq 3$ such that $s \geq 3$ and $d - 1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$. Then there exist a partition D and permutations $\gamma_1, \gamma_2 \in \Sigma_d$ such that the new collection of partitions $\hat{\mathcal{D}} = \{D, D_3, \dots, D_s\}$ satisfies the conditions $d - 1 < \nu(\hat{\mathcal{D}}) \equiv 0 \pmod{2}$ and $\gamma_1 \in D_1, \gamma_2 \in D_2, \gamma_1\gamma_2 \in D$.

Proof. Step 1. Without loss of generality we may and shall assume that $\sum_{i=3}^s \nu(D_i) > 1$. Suppose the contrary. Then $\sum_{i=3}^s \nu(D_i) = 1, s = 3$ and $D_3 = [2, 1 \dots, 1]$. In this case either $\nu(D_1) > 1$ or $\nu(D_2) > 1$, since otherwise $D_1 = D_2 = D_3 = [2, 1 \dots, 1]$, $\nu(\mathcal{D}) = 3$ is odd,

which is impossible. So, we can relabel the partitions in \mathcal{D} in such a way that $\nu(D_3) > 1$.

Step 2. Suppose $\nu(\mathcal{D}) = (d - 1) + 2q$ with $q > 0$. Then $\nu(D_1) + \nu(D_2) = (d - 1) + 2q - \sum_{i=3}^s \nu(D_i)$.

If $r := 2q - \sum_{i=3}^s \nu(D_i) \leq 0$, define $t := 1 - r \geq 1$ then $\nu(D_1) + \nu(D_2) = d - t$ and applying Lemma 4.1, there exist $\gamma_1 \in D_1, \gamma_2 \in D_2$ such that

$$\nu(\gamma_1\gamma_2) = d - t = (d - 1) + 2q - \sum_{i=3}^s \nu(D_i).$$

Let D be the partition determined by the cycle structure of $\gamma_1\gamma_2$. Then $\nu(D) + \sum_{i=3}^s \nu(D_i) = (d - 1) + 2q$ and

$$\nu(\hat{\mathcal{D}}) = \nu(\mathcal{D}),$$

where $\hat{\mathcal{D}} = \{D, D_3, \dots, D_s\}$. Moreover, since $d - 1 < \nu(\mathcal{D})$ then $d - 1 < \nu(\hat{\mathcal{D}})$.

If $r > 0$, let $k \in \{0, 1\}$ such that $r \equiv k \pmod{2}$. Applying Lemma 4.2, there are permutations $\gamma_1 \in D_1, \gamma_2 \in D_2$ such that $\nu(\gamma_1\gamma_2) = (d - 1) - k$. Let D be the partition determined by the cyclic structure of $\gamma_1\gamma_2$. Since $\nu(D) \equiv \nu(D_1) + \nu(D_2) \pmod{2}$ then

$$\nu(\hat{\mathcal{D}}) \equiv \nu(\mathcal{D}) \pmod{2},$$

where $\hat{\mathcal{D}} = \{D, D_3, \dots, D_s\}$. If $k = 0$ then

$$d - 1 < (d - 1) + \sum_{i=3}^s \nu(D_i) = \nu(D) + \sum_{i=3}^s \nu(D_i) = \nu(\hat{\mathcal{D}}),$$

where inequality is justified because $\sum_{i=3}^s \nu(D_i) > 0$ since $s \geq 3$. If $k = 1$ then $\nu(\widehat{\mathcal{D}}) = (d-1) - 1 + \sum_{i=3}^s \nu(D_i)$. By step 1, $\sum_{i=3}^s \nu(D_i) > 1$. Hence $d-1 < \nu(\widehat{\mathcal{D}})$. \square

Fundamental Lemma 4.4. *Let $\mathcal{D} = \{D_1, \dots, D_s\}$ be a collection of partitions of an odd integer $d \geq 3$ such that $d-1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$. Then there exist permutations $\sigma_i \in D_i$ for $1 \leq i \leq s$ such that $\sigma_1 \dots \sigma_s$ is a $(d-2)$ -cycle and the permutation group $G := \langle \sigma_1, \dots, \sigma_s \rangle \leq \Sigma_d$ is transitive. For any such a collection of permutations, the permutation group G is primitive.*

Proof. Step 1. Let us prove the first assertion of Lemma by induction on the number s of partitions of d in \mathcal{D} . First we observe that $s \geq 2$, since $\nu(\mathcal{D}) > d-1$.

For two partitions ($s = 2$), the first assertion of Lemma is proved in Theorem 3.1 (see the beginning of its proof). For the induction hypothesis, suppose that $k \geq 2$ and the assertion of Lemma holds for $s = k$, i.e. for every collection $\mathcal{D} = \{D_1, \dots, D_k\}$ of k partitions of d such that $d-1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$ there exist permutations $\sigma_i \in D_i$ for $1 \leq i \leq k$ such that $\sigma_1 \dots \sigma_k$ is a $(d-2)$ -cycle and the permutation group $\langle \sigma_1, \dots, \sigma_k \rangle \leq \Sigma_d$ is transitive. For the inductive step, consider a collection $\mathcal{D} = \{D_1, D_2, D_3, \dots, D_{k+1}\}$ of $k+1$ partitions of d such that $d-1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$. By Proposition 4.3, there exists a collection $\widehat{\mathcal{D}} = \{D, D_3, \dots, D_{k+1}\}$ of k partitions such that $d-1 < \nu(\widehat{\mathcal{D}}) \equiv 0 \pmod{2}$, moreover there exist permutations $\gamma_1 \in D_1, \gamma_2 \in D_2$ such that $\gamma_1 \gamma_2 \in D$.

Thus, by the induction hypothesis, there exist permutations $\sigma \in D, \sigma_i \in D_i$ for $i = 3, \dots, k+1$ in Σ_d , such that the permutation group $\widehat{G} = \langle \sigma, \sigma_3, \dots, \sigma_{k+1} \rangle$ is transitive and $\sigma \sigma_3 \dots \sigma_{k+1} \in [d-2, 1, 1]$. Since $\sigma \in D$ and $\gamma_1 \gamma_2 \in D$, it is clear that σ and $\gamma_1 \gamma_2$ are conjugate. Thus there exists $\lambda \in \Sigma_d$ such that $\sigma = \lambda \gamma_1 \gamma_2 \lambda^{-1}$. Define the permutation group $G = \langle \lambda \gamma_1 \lambda^{-1}, \lambda \gamma_2 \lambda^{-1}, \sigma_3, \dots, \sigma_{k+1} \rangle$ where the relation $\lambda \gamma_1 \gamma_2 \lambda^{-1} \sigma_3 \dots \sigma_{k+1} = \sigma \sigma_3 \dots \sigma_{k+1} \in [d-2, 1, 1]$ certainly holds. Since $\widehat{G} \leq G$ and \widehat{G} is transitive, the group G is transitive too. The induction step is completed.

Step 2. Let us prove the second assertion of Lemma. Since $\sigma_1 \dots \sigma_s$ is a $(d-2)$ -cycle and the group $G = \langle \sigma_1, \dots, \sigma_s \rangle$ is transitive, it follows from Example 1.2 that G is primitive. \square

Remark 4.5. *Lemma 4.4 is equivalent to the following assertion which is more convenient for studying branched coverings of the 2-sphere. Let $\mathcal{D} = \{D_1, \dots, D_s\}$ be a collection of partitions of an odd integer $d \geq 2$ such that $2d-2 \leq \nu(\mathcal{D}) \equiv 0 \pmod{2}$ and $D_1 = [d-2, 1, 1]$. Then there exist permutations $\sigma_i \in D_i$ for $1 \leq i \leq s$ such that $\sigma_1 \dots \sigma_s$ is the trivial permutation.*

Theorem 4.6. *Let \mathcal{D} be a collection of partitions of an odd integer d such that $d-1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$ (compare (4)). Then it can be realized as the branch*

datum of an indecomposable (and hence primitive) branched covering of degree d over the projective plane with a connected covering surface.

Proof. Since $\nu(\mathcal{D}) > d - 1$, we have $d \geq 2$. Since d is odd, we have $d \geq 3$. Let $\mathcal{D} = \{D_1, \dots, D_s\}$. By Fundamental Lemma 4.4, there exist permutations $\sigma_i \in D_i$ for $1 \leq i \leq s$ such that $\sigma_1 \dots \sigma_s$ is a $(d - 2)$ -cycle and the permutation group $G := \langle \sigma_1, \dots, \sigma_s \rangle \leq \Sigma_d$ is transitive and primitive.

Since $d - 2$ is odd and $\sigma_1 \dots \sigma_s$ is a $(d - 2)$ -cycle, we can define $\alpha := \sqrt{\sigma_1 \dots \sigma_s}$, so $G = \langle \alpha, \sigma_1, \dots, \sigma_s \rangle$ and $\sigma_1 \dots \sigma_s = \alpha^2$. Now we apply the Hurwitz approach to this collection of permutations. By Theorem 1.3 and Remark 1.4 we can realize G as the monodromy group of a branched covering $(M, \phi, \mathbb{R}P^2, B_\phi, d)$ whose branch datum is \mathcal{D} . Moreover, since the monodromy group G is transitive and primitive, by Proposition 2.1 the covering surface M is connected and the branched covering ϕ is indecomposable. \square

We observe that Fundamental Lemma 4.4 is in fact equivalent to the existence, for any collection $\mathcal{D} = \{D_1, \dots, D_s\}$ of partitions of an odd integer $d \geq 3$ with $D_1 = [d - 2, 1, 1]$, of a branched covering $(M, \phi, S^2, B_\phi, d)$ over S^2 whose branch datum is \mathcal{D} and the covering surface M is connected, provided that the Hurwitz conditions $2d - 2 \geq \nu(\mathcal{D}) \equiv 0 \pmod{2}$ hold. A similar result was obtained by R. Thom [12] about the existence, for any collection $\mathcal{D} = \{D_1, \dots, D_s\}$ of partitions of an integer $d \geq 3$ with $D_1 = [d]$, of a branched covering $(S^2, \phi, S^2, B_\phi, d)$ over S^2 whose branch datum is \mathcal{D} and the covering surface $M = S^2$, provided that the Hurwitz condition $\nu(\mathcal{D}) = 2d - 2$ holds. More precisely, we obtain from Fundamental Lemma 4.4 the following result that contributes to giving a partial solution of the realization problem for certain branch data [8].

Theorem 4.7. *Let $\mathcal{D} = \{D_1, \dots, D_s\}$ be a collection of partitions of an odd integer $d \geq 3$ such that $2d - 2 \leq \nu(\mathcal{D}) \equiv 0 \pmod{2}$ and $D_1 = [d - 2, 1, 1]$. Then \mathcal{D} can be realized as the branch datum of a branched covering of degree d over the 2-sphere with connected covering surface. Any such a branched covering is indecomposable (and hence primitive).*

Proof. Consider the collection $\hat{\mathcal{D}} := \mathcal{D} \setminus \{D_1\} = \{D_2, \dots, D_s\}$ of partitions of d . Since $\nu(\mathcal{D}) \geq 2d - 2$, we have $\nu(\hat{\mathcal{D}}) = \nu(\mathcal{D}) - \nu(D_1) = \nu(\mathcal{D}) - d + 3 \geq d + 1$.

By applying Fundamental Lemma 4.4 to $\hat{\mathcal{D}}$, we have that there exist permutations $\sigma_i \in D_i$ for $2 \leq i \leq s$ such that $\sigma_2 \dots \sigma_s$ is a $(d - 2)$ -cycle and the permutation group $G := \langle \sigma_2, \dots, \sigma_s \rangle \leq \Sigma_d$ is transitive and primitive.

Take the permutation $\sigma_1 := (\sigma_2 \dots \sigma_s)^{-1}$,

so $G = \langle \sigma_1, \dots, \sigma_s \rangle$ and $\sigma_1 \dots \sigma_s$ is the trivial permutation. Now we apply the Hurwitz approach to this collection of permutations. By Theorem 1.3 and analogue of Remark 1.4 for $N = S^2$, we can realize G as the monodromy group of

a branched covering $(M, \phi, S^2, B_\phi, d)$ whose branch datum is \mathcal{D} . Moreover, since the monodromy group G is transitive and primitive, by Proposition 2.1 the covering surface M is connected and the branched covering ϕ is indecomposable. \square

Clearly, Theorem 4.7 implies Fundamental Lemma 4.4.

5 Appendix

The table below is related with the second case in the proof of Lemma 3.4, for $d = 5, 7, 9, 11$. We list all the possibilities for $\mathcal{D} = \{D_1, D_2\}$, with $c_1 = d_1 = 3$ and $d - 1 < \nu(\mathcal{D}) \equiv 0 \pmod{2}$, as well their realization by permutations $\lambda \in D_1$, $\beta \in D_2$ such that $\alpha\beta$ is a $(d - 2)$ -cycle.

	t	d	$\mathcal{D} = \{D_1, D_2\}$	$\lambda \in D_1$	$\beta \in D_2$	$\lambda\beta \in [d - 2, 1, 1]$
1	2	5	$\{[3, 2], [3, 2]\}$	$(1\ 2\ 3)(4\ 5)$	$(5\ 4\ 1)(3\ 2)$	$(1\ 3\ 5)(2)(4)$
2	3	7	$\{[3, 3, 1], [3, 3, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7)$	$(3\ 2\ 4)(6\ 5\ 7)(1)$	$(1\ 4\ 7\ 6\ 3)(2)(5)$
3	3	7	$\{[3, 3, 1], [3, 2, 2]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7)$	$(3\ 2\ 4)(5\ 6)(7\ 1)$	$(1\ 4\ 6\ 3\ 7)(2)(5)$
4	3	7	$\{[3, 2, 2], [3, 2, 2]\}$	$(1\ 2\ 3)(4\ 5)(6\ 7)$	$(1\ 2)(5\ 4\ 6)(7\ 3)$	$(2\ 7\ 5\ 6\ 3)(1)(4)$
5	3	9	$\{[3, 3, 3], [3, 3, 1, 1, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$	$(3\ 2\ 4)(6\ 5\ 7)(1)(8)(9)$	$(1\ 4\ 7\ 8\ 9\ 6\ 3)(2)(5)$
6	3	9	$\{[3, 3, 3], [3, 2, 2, 1, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$	$(3\ 2\ 4)(6\ 5)(7\ 1)(8)(9)$	$(1\ 4\ 6\ 3\ 7\ 8\ 9)(2)(5)$
7	3	9	$\{[3, 3, 3], [3, 3, 3]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$	$(3\ 2\ 4)(6\ 5\ 7)(1\ 8\ 9)$	$(1\ 4\ 7\ 9\ 6\ 3\ 8)(2)(5)$
8	4	9	$\{[3, 3, 2, 1], [3, 3, 2, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8)(9)$	$(2\ 1\ 7)(6\ 5\ 9)(3\ 4)(8)$	$(2\ 4\ 9\ 6\ 3\ 7\ 8)(1)(5)$
9	4	9	$\{[3, 3, 2, 1], [3, 2, 2, 2]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8)(9)$	$(2\ 1\ 9)(4\ 7)(6\ 5)(3\ 8)$	$(2\ 8\ 4\ 6\ 7\ 3\ 9)(1)(5)$
10	4	9	$\{[3, 2, 2, 2], [3, 2, 2, 2]\}$	$(1\ 2\ 3)(4\ 5)(6\ 7)(8\ 9)$	$(5\ 4\ 6)(3\ 2)(1\ 8)(7\ 9)$	$(1\ 3\ 8\ 7\ 5\ 6\ 9)(2)(4)$
11	4	11	$\{[3, 3, 3, 2], [3, 3, 2, 1, 1, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11)$	$(5\ 4\ 1)(9\ 8\ 6)(2\ 10)(3)(11)(7)$	$(1\ 10\ 11\ 2\ 3\ 5\ 9\ 7\ 6)(4)(8)$
12	4	11	$\{[3, 3, 3, 2], [3, 2, 2, 2, 1, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11)$	$(3\ 4\ 7)(2\ 1)(6\ 5)(9\ 10)(8)(11)$	$(2\ 4\ 6\ 7\ 8\ 10\ 11\ 9\ 3)(1)(5)$
13	4	11	$\{[3, 3, 3, 2], [3, 3, 3, 2]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11)$	$(3\ 2\ 4)(6\ 5\ 7)(8\ 9\ 10)(1\ 11)$	$(1\ 4\ 7\ 9\ 6\ 3\ 11\ 8\ 10)(2)(5)$
14	5	11	$\{[3, 2, 2, 2, 2], [3, 2, 2, 2, 2]\}$	$(1\ 2\ 3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)$	$(5\ 4\ 6)(2\ 3)(1\ 8)(9\ 10)(7\ 11)$	$(1\ 3\ 8\ 10\ 7\ 5\ 6\ 11\ 9)(2)(4)$
15	5	11	$\{[3, 2, 2, 2, 2], [3, 3, 3, 1, 1]\}$	$(1\ 2\ 3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)$	$(5\ 4\ 6)(9\ 8\ 10)(1\ 7\ 11)(2)(3)$	$(1\ 2\ 3\ 7\ 5\ 6\ 11\ 9\ 10)(4)(8)$
16	5	11	$\{[3, 2, 2, 2, 2], [3, 3, 2, 2, 1]\}$	$(1\ 2\ 3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)$	$(5\ 4\ 6)(9\ 8\ 10)(11\ 1)(2\ 7)(3)$	$(1\ 7\ 5\ 6\ 2\ 3\ 11\ 9\ 10)(4)(8)$
17	5	11	$\{[3, 3, 3, 1, 1], [3, 3, 3, 1, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10)(11)$	$(3\ 2\ 4)(6\ 5\ 7)(9\ 10\ 11)(1)(8)$	$(1\ 4\ 7\ 8\ 10\ 11\ 9\ 6\ 3)(2)(5)$
18	5	11	$\{[3, 3, 3, 1, 1], [3, 3, 2, 2, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10)(11)$	$(3\ 2\ 4)(6\ 5\ 7)(9\ 10)(1\ 11)(8)$	$(1\ 4\ 7\ 8\ 10\ 9\ 6\ 3\ 11)(2)(5)$
19	5	11	$\{[3, 3, 2, 2, 1], [3, 3, 2, 2, 1]\}$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8)(9\ 10)(11)$	$(3\ 2\ 4)(8\ 7\ 6)(9\ 11)(1\ 10)(5)$	$(1\ 4\ 5\ 8\ 6\ 3\ 10\ 11\ 9)(2)(7)$

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Departament of Mathematics
 University Federal of São Carlos
 Rod. Washington Luis, Km. 235. C.P 676 - 13565-905 São Carlos, SP - Brazil
 nbedoya@dm.ufscar.br

Departament of Mathematics
 Institute of Mathematics and Statistic
 University of São Paulo
 Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brazil.
 dlgoncal@ime.usp.br

Department of Mathematics and Mechanics,
 Moscow State University
 Moscow 119992, Russia
 ekudr@gmx.de